A LOCALIZED ERDÖS-KAC THEOREM FOR $\omega_y(p+a)$

ANUP B. DIXIT, M. RAM MURTY

ABSTRACT. Let $\omega_y(n)$ denote the number of distinct prime divisors of n less than y. Suppose y_n is an increasing sequence of positive real numbers satisfying $\log y_n = o(\log\log n)$. In this paper, we prove an Erdös-Kac theorem for the distribution of $\omega_{y_n}(p+a)$, where p runs over all prime numbers and a is a fixed integer. We also highlight the connection between the distribution of $\omega_y(p-1)$ and Ihara's conjectures on Euler-Kronecker constants.

1. Introduction

Let $\omega(n)$ denote the number of distinct prime divisors of n. The distribution of $\omega(n)$ as we vary n, has been extensively studied in the last century. The average value of $\omega(n)$ for $n \leq x$ can be easily computed as

$$\frac{1}{x}\sum_{n\leq x}\omega(n) = \frac{1}{x}\sum_{p\leq x}\sum_{\substack{n\leq x,\\ p|n}} 1 = \frac{1}{x}\sum_{p\leq x}\left[\frac{x}{p}\right] = \log\log x + O(1)$$

as $x \to \infty$. In 1917, Hardy and Ramanujan [13] proved that the normal order of $\omega(n)$ is $\log\log n$. More precisely, for any $\varepsilon > 0$, as $x \to \infty$, we have

$$\#\left\{n \leq x \,\middle|\, n \text{ satisfies } |\omega(n) - \log\log n| > \varepsilon \log\log n\right\} = o(x). \tag{1}$$

A simplified proof of the Hardy-Ramanujan result was given by Turán [22] in 1934, by considering the second moment of $\omega(n)$. After Turán's paper appeared, M. Kac posed the question of finding the distribution of

$$\frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}, \tag{2}$$

as n varies and suggested that this distribution was perhaps Gaussian. This led to the famous Erdös-Kac theorem [7] which states that for any real numbers a, b

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n \leq x \,\middle|\, a \leq \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \leq b\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-t^2/2}\, dt.$$

Thus, the quantity in (2) has the standard normal distribution. The original proof of Erdös and Kac used Brun's sieve and the central limit theorem. Alternate proofs of the Erdös-Kac theorem were given using different methods by Selberg [20], Halberstam [11], Billingsley [1](using the method of moments which we adopt below) and Shapiro [21].

Date: September 26, 2022.

2010 *Mathematics Subject Classification.* 11A41, 11B50, 11Y35.

Key words and phrases. Erdös-Kac Theorem, Euler-Kronecker constant.

The research of the first author is partially supported by an Inspire Faculty fellowship. The research of the second author is partially supported by NSERC Discovery grant.

Localized Erdös-Kac theorem studies the distribution of $\omega_y(n)$, which denotes the number of prime divisors of n less than y. This line of study was initiated by the authors in [3], where they proved that the distribution of

$$\frac{\omega_{y_n}(n) - \log\log y_n}{\sqrt{\log\log y_n}} \tag{3}$$

is Gaussian as long as $\lim_{n\to\infty} \frac{\log y_n}{\log n} = 0$. In this paper, we study the distribution of $\omega_y(p+a)$, where p varies over prime numbers and a is some fixed integer. The distribution of $\omega(p+a)$ has been carefully studied in the last century. In fact, Erdös [6] proved that the normal order of $\omega(p-1)$ is $\log\log p$. Subsequently Haselgrove [14] showed that the normal order of $\omega(p+a)$ is $\log\log(p+a)$. Following further developments by Prachar [19], Halberstam [11] proved an Erdös-Kac type theorem for the distribution of $\omega(p+a)$. Our goal is to prove a localized Erdös-Kac theorem for the distribution of $\omega_y(p+a)$.

Let Ω be the set of positive integers and P_n be the probability measure placing mass $1/\pi(n)$ for each $\{2, 3, 5, \cdots, p_j\}$, where p_j denotes the largest prime $\le n$. We prove the following version of the Erdös-Kac theorem for $\omega_y(p+a)$.

Theorem 1.1. *Let y_n be an increasing sequence of real numbers satisfying $y_n \to \infty$ as $n \to \infty$ and suppose*

$$\lim_{n\to\infty} \frac{\log y_n}{\log\log(n)} = 0.$$

Then, for every pair of real numbers $\alpha and \beta$, we have

$$\lim_{n\to\infty} P_n\left(p \text{ prime: } \alpha \le \frac{\omega_{y_n}(p+a) - \log\log y_n}{\sqrt{\log\log y_n}} \le \beta\right) = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2}\, dt.$$

Remark. Note that there is a discrepancy in the rate at which y_n tends to infinity in (3) and Theorem 1.1. The heart of the proof of these results lies in exploiting the fact that primes behave like random variables. The closer it is to behaving like random variables, the better rate one can impose on y_n. This is captured in Lemma 2.3 of the next section. In the setting of all positive integers, one can say that primes are closer to behaving like random variables as opposed to integers of the form $\{p+a\}$, which is precisely the reason for this discrepancy.

2. A generalized central limit theorem

We elaborate in this section about a general method initiated in [3] that is applicable in a wider context. The proof of Theorem 1.1 relies on this method and the method of moments, stated below (see [2, pp. 312]).

Theorem 2.1 (Method of moments)**.** *Let μ be a probability measure on the line having finite moments*

$$\alpha_k = \int_{-\infty}^{\infty} x^k\, \mu(dx),$$

for all positive integers k. If the power series

$$\sum_{k=1}^{\infty} \frac{\alpha_k r^k}{k!}$$

has a positive radius of convergence, then μ is the only probability measure with moments $\alpha_1, \alpha_2, \cdots$.

The second ingredient is the central limit theorem. We state the Lyapunov central limit theorem below (see [2, pp. 342]).

Theorem 2.2 (Lyapunov central limit theorem)**.** *For $i \in \mathbb{N}$, let X_i be independent random variables, with mean μ_i and variance σ_i^2 respectively. Denote by $s_n^2 = \sum_{i=1}^n \sigma_i^2$. If for some $\delta > 0$, the Lyapunov condition*

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} E\left[|X_i - \mu_i|^{2+\delta} \right] = 0 \tag{4}$$

is satisfied, then we have

$$\frac{1}{s_n} \sum_{i=1}^{n} (X_i - \mu_i) \to N(0,1),$$

where $N(0,1)$ denotes the standard normal distribution, with mean 0 and variance 1.

The following is of independent interest and follows an idea from our earlier work [3]. We first prove a generalized central limit theorem, which will play a crucial role in the proof of Theorem 1.1.

Let f and g be non-decreasing functions on positive integers, such that as n tends to infinity, $f(n)$ and $g(n)$ tend to infinity and

$$\log f(n) = o(\log g(n)).$$

We prove the following.

Lemma 2.3. *For $i \in \mathbb{N}$, let X_i be independent random variables, taking bounded values and satisfying the Lyapunov condition (4) with mean μ_i and variance σ_i^2. Let Y_i be random variables, not necessarily independent such that*

$$E[X_{i_1} X_{i_2} \cdots X_{i_k}] = E[Y_{i_1} Y_{i_2} \cdots Y_{i_k}] + O\left(\frac{1}{g(n)} \right), \tag{5}$$

for $i_j \le f(n)$ for all $1 \le j \le k$. Then, as $n \to +\infty$ we have,

$$\frac{1}{s_n} \sum_{i=1}^{f(n)} (Y_i - \mu_i),$$

converges to the standard normal distribution N(0,1), where $s_n^2 = \sum_{i=1}^{f(n)} \sigma_i^2$.

Proof. Let $S_n = \sum_{j \le f(n)} X_j$ and $T_n = \sum_{j \le f(n)} Y_j$. Denote the mean and variance of S_n as c_n and s_n^2 respectively. As the Lyapunov condition is satisfied for S_n, by Theorem 2.2, we conclude that as n tends to infinity, $(S_n - c_n)/s_n$ converges to the standard normal distribution. Since X_n's are bounded, the method of moments applies here and from Theorem 2.1 we have that the r-th moment of $(S_n - c_n)/s_n$ converges to the r-th moment of the normal distribution, i.e.,

$$m_r = \lim_{n\to\infty} E\left[\left(\frac{S_n - c_n}{s_n} \right)^r \right]$$

for all r, where m_r denotes the r-th moment of the standard normal distribution.

Let d_n and r_n^2 denote the mean and variance of T_n respectively. By condition (5), we have

$$c_n = d_n + O(1) \quad \text{and} \quad s_n^2 = r_n^2 + O(1)$$

as n tends to infinity. Hence, to prove Theorem 2.3, it suffices to show that as $n \to \infty$

$$E\left[\left(\frac{S_n - c_n}{s_n} \right)^r \right] - E\left[\left(\frac{T_n - c_n}{s_n} \right)^r \right] \to 0 \tag{*}$$

for each r. We have

$$E[S_n^r] = \sum_{u=1}^{r} \sum{}' \frac{r!}{r_1! \cdots r_u!} \frac{1}{u!} \sum{}'' E[X_{i_1} \cdots X_{i_u}], \tag{6}$$

where $\sum{}'$ runs over tuples $(r_1, \cdots, r_u)$ satisfying $r_1 + \cdots + r_u = r$ and $\sum{}''$ is over tuples $(i_1, \cdots, i_u)$, where i_j's are distinct and not exceeding $f(n)$.

Similarly, we get

$$E[T_n^r] = \sum_{u=1}^{r} \sum{}' \frac{r!}{r_1! \cdots r_u!} \frac{1}{u!} \sum{}'' E[Y_{i_1} \cdots Y_{i_u}], \tag{7}$$

where $\sum'$ and $\sum''$ are as in (6). By (5), the summands in (6) and (7) differ by $O(1/g(n))$. Hence,

$$\left| E[S_n^r] - E[T_n^r] \right| \ll \frac{1}{g(n)} \left(\sum_{j \leq f(n)} 1 \right)^r = O\left(\frac{f(n)^r}{g(n)} \right).$$

Now we have

$$E[(S_n - c_n)^r] = \sum_{k=0}^{r} \binom{r}{k} E[S_n^k] (-c_n)^{r-k}.$$

Similarly, we obtain

$$E[(T_n - c_n)^r] = \sum_{k=0}^{r} \binom{r}{k} E[T_n^k] (-c_n)^{r-k}.$$

Comparing these expressions, we get

$$\left| E[(S_n - c_n)^r] - E[(T_n - c_n)^r] \right| \leq \sum_{k=0}^{r} \binom{r}{k} \frac{f(n)^k}{g(n)} c_n^{r-k} = \frac{(f(n) + c_n)^r}{g(n)}.$$

Since X_i's take bounded values, we have $c_n = O(f(n))$. Using the condition $\log f(n) = o(\log g(n))$, we conclude that

$$\lim_{n \to \infty} \frac{(f(n) + c_n)^r}{g(n)} = 0.$$

Dividing by s_n^r, we see that (*) follows. □

3. A localized Erdös-Kac theorem for $\omega_y(p + a)$

Proof of Theorem 1.1. Our method of proof follows Billingsley [1]. For a prime p, let

$$\delta_p(m) := \begin{cases} 1 & \text{if } p \mid m \\ 0 & \text{otherwise.} \end{cases}$$

Then, we find that

$$\omega_y(m) = \sum_{p \leq y} \delta_p(m).$$

We now invoke the Siegel-Walfisz theorem [23]. Let $\pi(x; q, a)$ denote the number of primes $\leq x$ such that $q \equiv a \bmod p$. Then, for $q \leq (\log x)^N$

$$\pi(x, q, a) = \frac{li(x)}{\phi(q)} + O\left(x \exp\left(-c_N \sqrt{\log x}\right) \right), \tag{8}$$

where $c_N > 0$ is a constant only depending on N.

Let $p_1, p_2, \cdots, p_u$ be a set of distinct primes $\leq y$. Using (8), as $n \to +\infty$

$$P_n\left[q \text{ prime}, \ q \leq n \,\middle|\, \delta_{p_1}(q+a) = \cdots = \delta_{p_u}(q+a) = 1\right] = \frac{1}{(p_1-1)(p_2-1)\cdots(p_u-1)} + O\left(\frac{1}{\log n}\right).$$

This indicates that under P_n, δ_{p_i}'s behave like independent random variables up to a small error. For a function f supported on primes, define

$$E_n[f] = \frac{1}{\pi(n)} \sum_{p \leq n} f(p).$$

For all primes p, let X_p be independent random variables taking values $\{0, 1\}$, satisfying

$$P[X_p = 1] = \frac{1}{p-1} \quad \text{and} \quad P[X_p = 0] = 1 - \frac{1}{p-1}.$$

If $p_1, \cdots, p_u$ are distinct, then we have

$$P\left[X_{p_1} = \cdots = X_{p_u} = 1\right] = \frac{1}{(p_1-1)(p_2-1)\cdots(p_u-1)}.$$

Let $S_n = \sum_{p \leq y_n} X_p$. The mean and variance of S_n are given by

$$c_n = \sum_{p \leq y_n} \frac{1}{p-1} = \log\log y_n + O(1)$$

and

$$s_n^2 = \sum_{p \leq y_n} \frac{1}{p-1}\left(1 - \frac{1}{p-1}\right) = \log\log y_n + O(1).$$

Since X_p's are independent, we have

$$E[X_{p_1} \cdots X_{p_u}] = \frac{1}{(p_1-1)\cdots(p_u-1)}.$$

We also have

$$E_n[\delta_{p_1} \cdots \delta_{p_u}] = \frac{1}{(p_1-1)\cdots(p_u-1)} + O\left(\frac{1}{\log n}\right).$$

Hence, we have

$$E[X_{p_1} \cdots X_{p_u}] - E_n[\delta_{p_1} \cdots \delta_{p_u}] = O\left(\frac{1}{\log n}\right)$$

for all $p_i \leq y_n$. The proof now follows from Lemma 2.3. $\square$

4. Ihara's conjecture and $\omega_y(p-1)$

There is an intricate connection between the distribution of $\omega_y(p-1)$ and Ihara's conjecture on Euler-Kronecker constants. This was also a motivation for Theorem 1.1, which is certainly an interesting result in its own respect. In this section, we state Ihara's conjecture and describe its connection to $\omega_y(p-1)$.

Let K be a number field and $\zeta_K(s)$ be the associated Dedekind zeta-function defined on the half-plane $\Re(s) > 1$ as

$$\zeta_K(s) := \sum_{\mathfrak{a}\subset\mathcal{O}_K} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}\subset\mathcal{O}_K} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1},$$

where $\mathfrak{a}$ runs over all non-zero integral ideals and $\mathfrak{p}$ runs over all non-zero prime ideals of the ring of integers $\mathcal{O}_K$.

The function $\zeta_K(s)$ has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$. If the Laurent expansion of $\zeta_K(s)$ near $s = 1$ is written in the form

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1),$$

then the Euler-Kronecker constant associated to K, introduced by Ihara [16] is defined as

$$\gamma_K := \frac{c_0}{c_{-1}}.$$

One could also view γ_K as the constant term in the Laurent expansion of the logarithmic derivative of $\zeta_K(s)$ at $s = 1$, i.e.,

$$-\frac{\zeta_K'}{\zeta_K}(s) = \frac{1}{s-1} - \gamma_K + O(s-1). \tag{9}$$

Note that when $K = \mathbb{Q}$, the Euler-Kronecker constant $\gamma_{\mathbb{Q}}$ is nothing but the Euler-Mascheroni constant γ. In [16], Ihara proved the following bounds for γ_K using Weil's explicit formula:

$$\gamma_K \le 2\log\log\sqrt{|d_K|} \quad \text{(under GRH)}$$
$$\gamma_K \ge -\log\sqrt{|d_K|} \quad \text{(unconditionally)},$$

where d_K denotes the discriminant of K over $\mathbb{Q}$. In [4], we indicated that there is no need to use the Weil explicit formula method to derive the upper bound and one can deduce an analogous upper bound directly and prove that

$$\gamma_K \le [2\log\log|d_K|]\left(1 + O\left(\frac{\log\log\log|d_K|}{\log\log|d_K|}\right)\right).$$

This essentially is the best known conditional upper bound for γ_K. However, Ihara noticed that these bounds are much sharper when $K = \mathbb{Q}(\zeta_m)$ is a cyclotomic field. Based on numerical computations for $m \le 8000$, Ihara [17] made the following conjectures. Henceforth, for a cyclotomic field $K = \mathbb{Q}(\zeta_m)$, the associated Euler-Kronecker constant will be denoted by γ_m.

Conjecture 1 (Ihara). *For $K = \mathbb{Q}(\zeta_m)$,*

a. $\gamma_m > 0$ for all m.
b. There exist positive constants c_1, c_2, both ≤ 2, such that for any $\varepsilon > 0$,

$$(c_1 - \varepsilon)\log m < \gamma_m < (c_2 + \varepsilon)\log m$$

for sufficiently large m. If m is a prime, one can choose $c_1 = 1/2$ and $c_2 = 3/2$.

In 2014, K. Ford, F. Luca and P. Moree [8] showed that the prime k-tuple conjecture, as formulated by Hardy and Littlewood, is incompatible with Ihara's conjectures.

A set of positive integers $\{a_1, a_2, \cdots, a_k\}$ is said to be admissible if collection of the form n and $a_i n + 1$, $1 \le i \le k$ have no fixed prime factor. The prime k-tuple conjecture states that for such an admissible set, the number of primes $n \le x$ for which $a_i n + 1$ are all primes is $\gg x/(\log x)^{k+1}$.

In fact, Ford, Luca and Moree showed that this conjecture implies $\gamma_q < 0$ infinitely often. By constructing a nice admissible set, they also explicitly produced a prime, namely $q = 964477901$, for which

$$\gamma_q = -0.18237\cdots$$

Furthermore, under the prime k-tuple conjecture, they showed that

$$\liminf_{q \to \infty} \frac{\gamma_q}{\log q} = -\infty.$$

In spite of this, it would seem that Conjecture 1(b) is not very far from the truth. In support of this, V. K. Murty [18] proved that

$$\sum_{q \sim Q,\, q \,\text{prime}} |\gamma_q| \ll \pi^*(Q) \log Q,$$

where $q \sim Q$ means $Q \le q \le 2Q$ and $\pi^*(Q)$ denotes the number of primes in this interval. E. Fouvry [9] generalized this to

$$\frac{1}{Q} \sum_{m \sim Q} \gamma_m = \log Q + O(\log\log Q),$$

where m runs over all positive integers in the interval and $Q \ge 3$. Both these results are quite deep and show that Conjecture 1(b) holds on average. In fact, assuming the Elliott-Halberstam conjecture, in [4], we prove that

$$\sum_{q \sim Q,\, q \,\text{prime}} |\gamma_q - \log q| = o(Q).$$

A similar result under the Elliot-Halberstam conjecture was also recently obtained by Hong, Ono and Zhang [15], i.e.,

$$\frac{1}{Q} \sum_{q \sim Q} |\gamma_q - \log q| = o(\log Q).$$

All the above results indicate that Conjecture 1(b) should fail very rarely, but perhaps infinitely often. However, it is still not known unconditionally whether $\gamma_m < 0$ for infinitely many positive integers m. In attempting to tackle this problem, we noticed that it is intricately connected to understanding the distribution of $\omega_y(p-1)$. We discuss this connection below.

In [4, Lemma 9.1], we prove that for any fixed $\delta > 0$, there is an $x_0(\delta) > 0$ such that for any $x > x_0(\delta)$ and any prime q satisfying $\log x > q^\delta$,

$$\gamma_q = -(q-1) \sum_{\substack{n \equiv 1 \bmod q \\ n \le x}} \frac{\Lambda(n)}{n} + \log x - \frac{\log q}{q-1} + O\left((\log x)^{\frac{1}{2} + \frac{1}{\delta}} \exp\left(-c\sqrt{\log x} \right) \right). \tag{10}$$

Note that the summation on the right hand side is bounded over higher prime powers. Indeed, for any $x > 1$ not a prime power, and noting that f_p is the order of $p \pmod q$, we have

$$\begin{aligned} S(q,x) := (q-1) \sum_{\substack{n\equiv 1 \bmod q \\ n\leq x}} \frac{\Lambda(n)}{n} - \sum_{\substack{p\equiv 1 \bmod q \\ p\leq x}} \frac{(q-1)\log p}{p} \\ = \sum_{p\equiv 1 \bmod q} \sum_{\substack{l\geq 2 \\ p^l\leq x}} \frac{(q-1)\log p}{p^l} + \sum_{p\not\equiv 1 \bmod q} \sum_{\substack{l\geq 1 \\ p^l\leq x}} \frac{(q-1)\log p}{p^{l f_p}} \\ \leq \sum_{\substack{p\equiv 1 \bmod q \\ p\leq x}} \frac{(q-1)\log p}{p(p-1)} + \sum_{\substack{p\not\equiv 1 \bmod q \\ p\leq x}} \frac{(q-1)\log p}{(p^{f_p}-1)} + O\left(\frac{\log q}{q}\right). \end{aligned}$$

Let us note that in the last sum, $f_p \geq 2$ because $p \not\equiv 1 \bmod q$. Since

$$\sum_{\substack{p\equiv 1 \bmod q \\ p\leq x}} \frac{\log p}{p(p-1)} \leq \sum_{t=1}^{\infty} \frac{\log qt}{(qt+1)qt} \ll \frac{\log q}{q^2},$$

the first term on the right hand side is $O((\log q)/q)$, where the implied constant is independent of q. For $f_p > 1$, we have $q \mid (p^{f_p} - 1) = (p-1)(p^{f_p-1} + p^{f_p-2} + \cdots + 1)$. Since $q \nmid (p-1)$, we get $q \mid (p^{f_p}-1)/(p-1)$ and hence $q \leq p^{f_p}/(p-1) \leq 2\,p^{f_p-1}$. Thus, the third term on the right hand side is

$$\sum_{\substack{p\not\equiv 1 \bmod q \\ p\leq x}} \frac{(q-1)\log p}{p^{f_p-1}\,(p^{f_p}-1)} \ll \sum_{n=1}^{\infty} \frac{\log n}{n^2} \ll 1.$$

Hence, we have

$$S(q,x) = O(1).$$

With this observation, applying (10) and summing over primes q up to y, we obtain

$$\begin{aligned} \sum_{q\leq y} \frac{\gamma_q}{q-1} = \log x \sum_{q\leq y} \frac{1}{q-1} - \sum_{q\leq y} \sum_{\substack{p\leq x \\ p\equiv 1 \bmod q}} \frac{\log p}{p} + O(\log\log y) \\ = \log x \sum_{q\leq y} \frac{1}{q-1} - \sum_{p\leq x} \frac{\log p}{p}\, \omega_y(p-1) + O(\log\log y). \qquad (11) \end{aligned}$$

Recall that,

$$\sum_{q\leq y} \frac{1}{q-1} = \log\log y + O(1),$$

with the error term ≤ 2. Now, using Chebychev's theorem

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1),$$

in equation (11), we obtain for any $x > 1$ not a prime power,

$$\sum_{q\leq y} \frac{\gamma_q}{q-1} = -\sum_{p\leq x} \frac{\log p}{p} (\omega_y(p-1) - \log\log y) + c\log x + O(\log\log y),$$

where $c < 2$. Hence, it is clear that the oscillation in $\omega_y(p-1) - \log\log y$ holds the key to the distribution of γ_q. For instance, to show that $\gamma_q < 0$ infinitely often, it suffices to prove that $\omega_y(p-1) - \log\log y$ oscillates in such a way that the first summand in the above sum is $< -2\log x$, infinitely often. On the other hand, if $\gamma_q > 0$ for all q sufficiently large, above sum should tally with [4, Theorem 1.4].

Remark. This idea can also be used in the study of γ_m, when m is not a prime. In this regard, Ihara's Conjecture 1(a) asserts that $\gamma_m > 0$. Thus, to show that Ihara's conjecture fails infinitely often, one would like to show that there are infinitely many integers m such that $\gamma_m < 0$. Using the explicit formula obtained by Gun, Murty and Rath in [10], we obtain

$$\begin{aligned}\sum_{m\le y}\frac{\gamma_m}{\phi(m)} &= \log x\sum_{m\le y}\frac{1}{\phi(m)} - \sum_{m\le y}\sum_{\substack{p\le x\\ p\equiv 1 \bmod m}}\frac{\log p}{p} + O(\log y)\\ &= \log x\sum_{m\le y}\frac{1}{\phi(m)} - \sum_{p\le x}\frac{\log p}{p}\, d_y(p-1) + O(\log y),\end{aligned}$$

where $d_y(n)$ counts the number of divisors of n less than y. We hope that this treatment would provide an alternate approach towards this problem.

5. **Concluding Remarks**

The above method can be used to establish a localized Erdös-Kac theorem for more general functions of the form $\omega_y(f(p))$. In the proof above, a key role was played by the Siegel-Walfisz theorem. In fact, for a function f, if we can write

$$\sum_{\substack{p\le n\\ f(p)\equiv 0 \bmod q_1\cdots q_u}} 1 = \text{ (main term) } + \text{ (error term) },$$

for all primes $q_1,\cdots,q_u$ sufficiently smaller than n, we can get a localized Erdös-Kac theorem for $\omega_y(f(p))$, provided the error term is sufficiently small.

Acknowledgements

We thank the referee for helpful comments on an earlier version of this paper.

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Institute of Mathematical Sciences (HBNI), CIT Campus Taramani, Chennai, India, 600113
Email address: anupdixit@imsc.res.in

Department of Mathematics and Statistics, Queen's University, Kingston, Canada, ON K7L 3N6.
Email address: murty@mast.queensu.ca